MATH 201: PRODUCT AND QUOTIENT RULES FOR DERIVATIVES
DUE WEDNESDAY, 17 OCT 2007

Theorem 83.1. *If f and g are differentiable real-valued functions defined on an open interval $I \subseteq \mathbb{R}$, then*

(a) *The function $f \cdot g$ is differentiable, and $(f \cdot g)' = f' \cdot g + f \cdot g'$.*

(b) *The function $\frac{f}{g}$ is differentiable, and $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$ at all points x where $g(x) \neq 0$.*

Proof. **Your assignment: Prove the quotient rule**.

We prove the product rule.

Let f and g be differentiable real-valued functions defined on an open interval $I \subseteq \mathbb{R}$. Let $a \in I$. We prove that the function $f \cdot g$ is differentiable at a and $(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$.

By Lagrange's Theorem, there are functions v, w defined on I satisfying $\lim_{x \to a} v(x) = \lim_{x \to a} w(x) = 0$ so that for all $x \in I$,

$$f(x) = f(a) + (f'(a) + v(x))(x - a) \tag{1}$$

$$g(x) = g(a) + (g'(a) + w(x))(x - a). \tag{2}$$

With the aim of using Lagrange's Theorem to prove that fg is differentiable, we multiply equations (1) and (2) to find

$$\begin{aligned} f(x)g(x) = & f(a)g(a)+ \\ & + f(a)g'(a) + f(a)w(x)(x-a) \\ & + f'(a)g(a) + g(a)v(x)(x-a) \\ & + (f'(a) + v(x))(x-a)(g'(a) + w(x))(x-a) \\ = & f(a)g(a) + (f(a)g'(a) + f'(a)g(a))(x-a) \\ & + (f(a)w(x) + g(a)v(x))(x-a) \\ & + (f'(a) + v(x))(g'(a) + w(x))(x-a)^2. \end{aligned} \tag{3}$$

Let

$$M = f(a)g'(a) + f'(a)g(a) \tag{4}$$

and

$$
(5) \qquad W(x) = f(a)w(x) + g(a)v(x) + (f'(a) + v(x))(g'(a) + w(x))(x - a).
$$

Then equation (3) may be reorganized into

$$
f(x)g(x) = f(a)g(a) + (M + W(x))(x - a)
$$

with the constant M given by equation (4) and the function W defined by equation (5).

We find by repeated application of the rules for limit of sum and product that

$$
\lim_{x \to a} W(x) = f(a) \cdot 0 + g(a) \cdot 0 + (f'(a) + 0)(g'(a) + 0) \cdot 0 \qquad = 0.
$$

We conclude by Lagrange's Theorem that $f \cdot g$ is differentiable at a and its derivative there is $M = f(a)g'(a) + f'(a)g(a)$. □

► When you form the difference quotient for f/g, there will be a $g(x+h)$ in a denominator. This is allowed only if $g(x+h) \neq 0$, but we are assuming only that $g(x) \neq 0$. To show that $g(x+h) \neq 0$ if h is small enough, reason as follows:

(a) The function g is continuous at x. (What hypothesis implies this?)
(b) $\lim_{h \to 0} g(x+h) = g(x)$ (Why?)
(c) Since $g(x) \neq 0$, Lemma 73.2 on nonzero limits shows that there exists a $\delta_0 > 0$ such that $|h| < \delta_0$ implies $|g(x+h)| \geq \frac{1}{2}|g(x)|$.