Mathematics 201: Limits I

All of analysis is founded on the concept of *limit*.

Definition 71.1 (Limit). *Let* $I \subseteq \mathbf{R}$ *be an open interval, let* $a \in I$*, and let* f *be a real-valued function defined on* I *except possibly at* a*. Let* $L \in \mathbf{R}$. *We say that* the limit of $f(x)$ as x approaches a is equal to L *if for every* $\epsilon > 0$ *there exists a* $\delta > 0$ *so that for all* $x \in I$, $0 < |x - a| < \delta$ *implies* $|f(x) - L| < \epsilon$.

When the limit of $f(x)$ as x approaches a is equal to L, we write

$$\lim_{x \to a} f(x) = L$$

and say $f(x)$ approaches L as x approaches a.

Observe that the set defined by the condition $0 < |x - a| < \delta$ is the union of two open intervals:

$$\{x \in \mathbf{R} \mid 0 < |x - a| < \delta\} = (a - \delta, a) \cup (a, a + \delta).$$

Notice too a peculiar thing about the definition of limit: even though we express the existence of the limit by saying "$f(x)$ approaches L," in fact there is no "approaching" of anything! The whole situation is completely static. The definition says only that for every positive ϵ a positive δ exists such that the part of the graph of f in the vertical strip

$$(a - \delta, a) \cup (a, a + \delta) \times \mathbf{R}$$

is entirely contained in the two open rectangles

$$((a - \delta, a) \times (L - \epsilon, L + \epsilon)) \bigcup ((a, a + \delta) \times (L - \epsilon, L + \epsilon)).$$

In-class Exercise:

(a) Complete the sentence, "The limit of $f(x)$ as x approaches a is equal to L provided ..." in symbolic form. How many nested quantifiers does it use?

(b) State a useful negation of your answer to (a) in symbolic form. That is, complete the sentence, "The limit of $f(x)$ as x approaches a is not equal to L provided ...".

(c) Complete the sentence, "The limit of $f(x)$ as x approaches a is not equal to L provided ..." in English.

To prepare for working with limits, review our earlier work on absolute value and the triangle inequality (§3.4 of the textbook).

Theorem 71.1 (Uniqueness of limits). *Let I be an open interval, let $a \in I$, and let f be a real-valued function defined on I except possibly at a. If $L, M \in \mathbf{R}$ satisfy $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$ then $L = M$.*

Sketch for a proof by contradiction: Suppose that $L \neq M$. Then one of them is smaller than the other. Call the smaller one L and larger one M so that $L < M$.

Let $\epsilon = \frac{1}{2}(M - L) > 0$. Since $\lim_{x\to a} f(x) = L$, then by the definition of limit there exists a $\delta_1 > 0$ such that $0 < |x - a| < \delta_1$ implies

$$|f(x) - L| < \epsilon. \tag{1}$$

Similarly, there exists a $\delta_2 > 0$ such that $0 < |x - a| < \delta_2$ implies

$$|f(x) - M| < \epsilon. \tag{2}$$

Now let $\delta = \min(\delta_1, \delta_2)$. Then $0 < |x - a| < \delta$ implies *both* inequalities (1) and (2). Unpacking them using Theorem 3.25.1, we find that $0 < |x - a| < \delta$ implies

$$L - \epsilon < f(x) < L + \epsilon, \tag{3}$$

$$\text{and} \qquad M - \epsilon < f(x) < M + \epsilon. \tag{4}$$

By our choice of ϵ, $L + \epsilon = M - \epsilon = \frac{1}{2}(L + M)$. Combining the right inequality in (3) with the left inequality in (4) we have, for all x satisfying $0 < |x - a| < \delta$,

$$f(x) < \frac{1}{2}(L + M) < f(x),$$

and this contradicts trichotomy.

Theorem 71.2 (Linearity of limits). *If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ and $c \in \mathbf{R}$, then*

(a) $\lim\limits_{x\to a} c \cdot f(x) = c \cdot L$.

(b) $\lim_{x\to a} f(x) + g(x) = L + M$.

Prove this Theorem. For part (a) it may work best to consider separately the cases $c = 0$ and $c \neq 0$. When $c \neq 0$ you can make $|c \cdot f(x) - c \cdot L| < \epsilon$ by making $|f(x) - L| < \epsilon/|c|$.

To prove part (b) you must determine how close x must be to a to enforce $|f(x) + g(x) - (L + M)| < \epsilon$. Use the triangle inequality to show that this will be true if

$$|f(x) - L| + |g(x) - M| < \epsilon, \tag{5}$$

and inequality (5) is true if both $|f(x) - L| < \epsilon/2$ and $|g(x) - M| < \epsilon/2$.