Mathematics 201: Ordered Fields IV

Theorem 41.1. *The complex field* $\mathbf{C}$ *is not an ordered field.*

Proof. We use a proof by contradiction to show that $\mathbf{C}$ is not an ordered field.

Assume that there is a relation "$<$" that makes $\mathbf{C}$ an ordered field. In every ordered field we have $1 > 0$ by the Corollary to Theorem 32.2; therefore $-1 < 0$ by Theorem 23.1.

Consider now the imaginary unit $i \in \mathbf{C}$. Since $i \neq 0$, it follows by Theorem 32.2 that $i^2 = -1 > 0$.

Thus, if $\mathbf{C}$ were an ordered field, we would have both $-1 > 0$ and $-1 < 0$, contradicting trichotomy. Therefore the complex field $\mathbf{C}$ is *not* an ordered field, and the proof is complete. □

Theorem 41.2. *The finite field* $\mathbf{Z}_7$ *is not an ordered field.*

Proof. We use a proof by contradiction to show that $\mathbf{Z}_7$ is not an ordered field. Suppose there is a relation "$<$" making $\mathbf{Z}_7$ an ordered field. Then, as in the proof of Theorem 41.1, we have $6 < 0$ in $\mathbf{Z}_7$, since 6 is the additive inverse of 1 in $\mathbf{Z}_7$.

Recall again that $1 > 0$ by the Corollary to Theorem 32.2. Adding 1 to this inequality gives $2 > 1$, so by the transitive property we have $2 > 0$. Adding 2 to this last inequality yields $4 > 2$ and thus $4 > 0$. Add 2 once again to obtain $6 > 2$, and so, finally, $6 > 0$.

We have proved that if $\mathbf{Z}_7$ were an ordered field, both inequalities $6 < 0$ and $6 > 0$ would be true, contradicting trichotomy. Therefore $\mathbf{Z}_7$ is not an ordered field, and the proof is complete. □