Mathematics 201: Ordered Fields II

Recall that in the field $\mathbf{F}_7$ the product $6 \times 6 = 1$, because in the integers 6×6 has the remainder 1 mod 7:

$$6 \times 6 = 7 \times 5 + 1.$$

However, in $\mathbf{F}_7$, $6 = -1$, so $6 \times 6 = 1$ can also be understood to mean $(-1) \times (-1) = 1$.

Assignment. Use Theorem 12.3 to prove that this last equation is valid in *every* field:

Theorem 32.1. *Let k be* any *field, and write -1 for the additive inverse of $1 \in k$. Then*

$$(-1) \times (-1) = 1. \qquad (1)$$

Continuing now with *ordered* fields, we prove two fundamental facts: the square of any nonzero element is positive, and 1 is positive.

Theorem 32.2. *Let k be an ordered field, $a \in k$, $a \neq 0$. Then $a^2 > 0$.*

Proof. Let k be an ordered field, $a \in k$, $a \neq 0$. We prove directly that $a^2 > 0$. By trichotomy, since $a \neq 0$, either $a > 0$ or $a < 0$. Therefore in the product $a^2 = a \times a$, both factors have the same sign, so the product is positive by Theorem 23.2. □

Corollary. *In any ordered field, $1 > 0$.*

Proof. Since $1 \neq 0$ (definition of field) and $1 = 1 \times 1 = 1^2$ (identity for multiplication), it follows from Theorem 32.2 that $1 > 0$. □