TANGENT LINES IN POLAR COORDINATES

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1. Introduction

In this article we develop the polar coordinate form of the equation for the tangent line to a curve when the curve is given in polar coordinates. We conclude with several examples in which the polar coordinate form of the tangent line is especially simple and yields geometric insight.

2. Tangent Lines

Let (r, θ) denote the usual polar coordinates of points in the plane. Our task is to find an equation for the tangent line to a curve

$$r = f(\theta) \tag{1}$$

at a point $(f(\theta_1), \theta_1)$.

Recall that in polar coordinates the equation of a line has one of two forms. If the line passes through the pole, its equation has the form

$$\theta = \theta_0, \tag{2}$$

while if the line does not pass through the pole it has an equation

$$r = \frac{d}{\cos(\theta - \theta_0)}. \tag{3}$$

Here θ_0 is the polar angle of the line's normal, and d is the distance along the normal from the pole to the line.

We begin by rewriting the polar equation (1) in parametric form as

$$\begin{aligned} x &= f(\theta)\cos\theta \\ y &= f(\theta)\sin\theta. \end{aligned}$$

Next, in the usual way, we use the Chain Rule and the Inverse Function Theorem to express the derivative dy/dx as (we write $f' = df/d\theta$)

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'\sin\theta + f\cos\theta}{f'\cos\theta - f\sin\theta}. \tag{4}$$

First, if $f(\theta_1) = 0$, that is, if the curve passes through the pole at $\theta = \theta_1$, then, simplifying (4), we find that the slope of the tangent line at $(f(\theta_1), \theta_1) = (0, \theta_1)$ is

$$\frac{dy}{dx} = \frac{\sin\theta_1}{\cos\theta_1} = \tan\theta_1. \tag{5}$$

Thus, at a point where $f(\theta_1) = 0$ the tangent line has the polar equation

$$\theta = \theta_1. \tag{6}$$

We now consider the case $f(\theta_1) \neq 0$, and look for an equation for the tangent line in the form of Equation (3). In that equation the angle θ_0 is the direction of the normal to the curve $f = f(\theta)$ at the point (r_1, θ_1). The normal slope is the negative reciprocal of the tangent slope (4), so we calculate

$$\tan\theta_0 = -\frac{dx/d\theta}{dy/d\theta} = -\frac{f'\cos\theta - f\sin\theta}{f'\sin\theta + f\cos\theta}$$

and, dividing numerator and denominator by $f\cos\theta$,

$$\tan\theta_0 = \frac{\tan\theta - f'/f}{1 + \frac{f'}{f}\tan\theta} \tag{7}$$

We recognize the addition formula for the tangent function. Setting

$$\psi = \arctan(f'/f)$$

we see from Equation (7) that $\tan\theta_0 = \tan(\theta_1 - \psi)$, or

$$\theta_0 = \theta_1 - \psi. \tag{8}$$

It remains to find d in Equation (3). Since the point (r_1, θ_1) of the curve is on the tangent line, we have

$$\begin{aligned} d &= r_1\cos(\theta_1 - \theta_0) \\ &= r_1\cos\psi \end{aligned} \tag{9}$$

To summarize: on a polar-coordinate curve $r = f(\theta)$, at a point (r_1, θ_1) with $r_1 = f(\theta_1) \neq 0$, the polar equation of the tangent line is

$$r = r_1\cos\psi\sec(\theta - \theta_0). \tag{10}$$

In this equation we have

$$\begin{aligned} r_1 &= f(\theta_1) \\ r_1' &= f'(\theta_1) \\ \psi &= \arctan(r_1'/r_1) \\ \theta_0 &= \theta_1 - \psi. \end{aligned}$$

Remark. The tangent line to a curve is horizontal when the normal line is vertical, and *vice versa*. Since θ_0 is the direction of the normal line, points where the tangent line is horizontal or vertical can easily be found.

3. Examples

In this section we give several examples of tangent lines with a simple form.

3.1. **Circle, center at the pole.**

A circle centered at the pole has an equation

$$r = a$$

with constant $a > 0$. Then $r' = 0$ so $\psi = 0$ and $\theta_0 = \theta_1$. The tangent line at (a, θ_1) is

$$r = a\sec(\theta - \theta_1).$$

This reflects the theorem from geometry that the tangent line to a circle is perpendicular to the radius drawn to the point of contact.

3.2. **Circle, diameter on polar axis.**

A circle whose diameter lies on the polar axis with one end at the pole has the equation

$$r = 2a\cos\theta$$

with a constant $a > 0$. Thus $r' = -2a\sin\theta$ and

$$\psi = \arctan\left(\frac{-2a\sin\theta_1}{2a\cos\theta_1}\right) = -\theta_1.$$

The equation of the tangent line at $(2a\cos(\theta_1), \theta_1)$ is

$$r = 2a\cos^2\theta_1 \sec(\theta - 2\theta_1).$$

3.3. **Parabola.** A parabola with focus at the pole, opening to the left, has equation

$$r = \frac{d}{1 + \cos\theta}.$$

Now $r' = -d(1 + \cos\theta)^{-2}(-\sin\theta)$, so

$$\begin{aligned}\tan\psi &= \frac{d\sin\theta_1}{(1+\cos\theta_1)^2}\frac{1+\cos\theta_1}{d} \\ &= \frac{\sin\theta_1}{1+\cos\theta_1} \\ &= \frac{2\sin\frac{1}{2}\theta_1\cos\frac{1}{2}\theta_1}{1 + 2\cos^2\frac{1}{2}\theta_1 - 1} \\ &= \tan\frac{1}{2}\theta_1.\end{aligned}$$

Thus $\psi = \theta_1/2$.[1] The tangent line to the parabola is

$$r = \frac{d}{1+\cos\theta_1}\frac{\cos\frac{1}{2}\theta_1}{\cos(\theta - \frac{1}{2}\theta_1)}.$$

3.4. **Cardioid.** Consider the cardioid with equation

$$r = a(1 + \cos\theta).$$

In this case we have $r' = -a\sin\theta$ and so

$$\tan\psi = \frac{-a\sin\theta_1}{a(1+\cos\theta_1)} = -\frac{1}{2}\theta_1$$

as for the parabola. Thus $\theta_0 = \frac{3}{2}\theta_1$ and the tangent line's equation is

$$r = \frac{(1+\cos\theta_1)\cos\frac{1}{2}\theta_1}{\cos(\theta - \frac{3}{2}\theta_1)}.$$

3.5. **Lemniscate.** A lemniscate has an equation

$$r^2 = 2a\cos 2\theta.$$

By implicit differentiation we find

$$2rr' = -4a\sin 2\theta,$$

and, dividing by $2r^2 = 4a\cos 2\theta$, we obtain

$$\frac{r'}{r} = -\frac{4a\sin 2\theta}{4a\cos 2\theta} = -\tan 2\theta.$$

[1]This fact leads to an easy solution to Problem 31 from §10.1 of Varberg: The tangents to a parabola at the ends of a focal chord are perpendicular to each other.

This gives the simple result

$$\psi = -2\theta_1,$$

and so $\theta_0 = \theta_1 - \psi = 3\theta_1$. The tangent line is

$$r = \frac{r_1 \cos 2\theta_1}{\cos(\theta - 3\theta_1)}.$$

3.6. **Logarithmic Spiral.** A logarithmic spiral has the polar equation

$$r = e^{a\theta}.$$

It follows very simply that $\tan\psi = r'/r = a$, so then $\theta_0 = \theta_1 - a$. This demonstrates that the normal line to a logarithmic spiral makes a fixed angle with the line to the pole. The equation of the tangent line is

$$r = \frac{e^{a\theta_1}}{\sqrt{1+a^2}\cos(\theta - \theta_1 + a)}.$$