Mathematics 165T Problem Set 6–Supplement

Outline of the Proof of Lemma 1.

Lemma 1 *Let f be differentiable on an open interval containing $[a, b]$ with $a < b$. For any constant K such that $f(b) - f(a) \geq K(b - a)$ there is a point x in $[a, b]$ such that $f'(x) \geq K$.*

Step I: Define inductively a sequence of intervals $I_n = [a_n, b_n]$, $n \geq 0$ satisfying

(i) $I_{n+1} \subset I_n$

(ii) $b_n - a_n = 2^{-n}(b - a)$

(iii) $\dfrac{f(b_n) - f(a_n)}{b_n - a_n} \geq K$.

By the completeness axiom, there is exactly one point common to all the I_n; call it x. Evidently $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x$.

Step II: Show that

$$\lim_{n\to\infty} \frac{f(b_n) - f(a_n)}{b_n - a_n} = f'(x). \tag{1}$$

It then follows from (iii) that $f'(x) \geq K$.