Mathematics 165T Problem Set 6

Week of March 19–March 23, 2001

We'll start with the justification of the Calculus procedure for finding a maximum or a minimum: "take the derivative and set it to zero."

Theorem 1 *If a function f is differentiable at a point a and has a local maximum or minimum at a, then $f'(a) = 0$.*

To prove the Mean Value Theorem for derivatives, we need the completeness of the real number system.

Axiom 1 (Completeness) *Let $(I_n = [a_n, b_n])_{n=1}^{\infty}$ be a sequence of nested ($I_n \supset I_{n+1}$) closed intervals with $\lim_{n\to\infty} b_n - a_n = 0$. Then there is exactly one point that belongs to all of the I_n.*

Lemma 1 *Let f be differentiable on an open interval containing $[a, b]$ with $a < b$. For any constant K such that $f(b) - f(a) \geq K(b - a)$ there is a point x in $[a, b]$ such that $f'(x) \geq K$.*

Corollary 1 *The Lemma is also valid if "$\geq$" is replaced by "$\leq$" throughout.*

Theorem 2 (Mean Value Theorem for Derivatives) *Let f be a differentiable function on an interval $[a, b]$ with*

$$m \leq f'(x) \leq M$$

for all x in $[a, b]$. Then

$$f(a) + m(b - a) \leq f(b) \leq f(a) + M(b - a).$$