1. The numbers $\varphi_1$, $\varphi_2$, $\varphi_3$, ... are defined by

$$\varphi_1 = 1, \quad \varphi_{k+1} = \frac{1}{1 + \varphi_k}$$

so that $\varphi_1 = 1$, $\varphi_2 = \frac{1}{2}$, $\varphi_3 = \frac{2}{3}$, $\varphi_4 = \frac{3}{5}$, and so on.

Calculate $\varphi_1, \ldots, \varphi_{20}$, expressing each one as a ratio of integers.

2. Verify the following facts for all $k \geq 1$ using $F(x) = x^2 + x - 1$.

$$\varphi_{k+2} = 1 + \frac{\varphi_k}{2 + \varphi_k}$$

$$\varphi_{k+2} - \varphi_k = -\frac{F(\varphi_k)}{2 + \varphi_k}$$

$$\varphi_{k+1} - \varphi_k = \frac{\varphi_k - \varphi_{k-1}}{(1 + \varphi_k)(1 + \varphi_{k-1})}$$

$$F(\varphi_{k+1}) = \frac{F(\varphi_k)}{(1 + \varphi_k)^2}$$

$$F(\varphi_{k+2}) = \frac{F(\varphi_k)}{(2 + \varphi_k)^2}$$

3. Prove the following statements, using the facts stated above as needed.

(a) The odd-numbered $\varphi_k$ form a decreasing sequence: $\varphi_1 > \varphi_3 > \ldots > \varphi_{2m-1} > \ldots$

(b) The even-numbered $\varphi_k$ form an increasing sequence: $\varphi_2 < \varphi_4 < \ldots < \varphi_{2n} < \ldots$

(c) Every odd-numbered $\varphi_{2m+1}$ is larger than every even-numbered $\varphi_{2n}$.

(d) $\lim_{k \to \infty} |\varphi_{k+1} - \varphi_k| = 0$.

(e) There is exactly one number $\varphi$ that is larger than all the even-numbered $\varphi_k$ and smaller than all the odd-numbered $\varphi_k$, and $F(\varphi) = 0$. 