The Parabolic Burning-Mirror

1 The Parabola as a Locus

A parabolic mirror reflects light rays incident parallel to its axis into a single point called the focus. This focusing property of the parabola was known to the ancient Greek geometers. In this project you will study the tangent and normal lines to the parabola and demonstrate the focusing property.

The equation of a parabola, with vertex at the origin, having the $x$-axis for its axis of symmetry can be written

$$y^2 = 4px. \tag{1}$$

**Problem 1.** Prove using analytic geometry that every point on the parabola is equidistant from the point $(p,0)$ (the focus) and the line $x = -p$ (the directrix).

2 The Tangent and Normal

Let $P(x_0,y_0)$ be an arbitrary point on the parabola (1).

**Problem 2.** Use Calculus and analytic geometry to find the point where the line tangent to the parabola at $P$ intercepts the axis of the parabola. Use your result to formulate a rule not requiring calculus for drawing the tangent line to a parabola at an arbitrary point.

**Problem 3.** The line normal to the parabola at $P$ is the line through $P$ that is perpendicular to the tangent line to the parabola at $P$. Find the point where the line normal to the parabola at $P$ intercepts the axis of the parabola. Use your result to formulate a purely geometric rule for drawing the normal line to the parabola at an arbitrary point.
3 The Reflection Property

The path of a light ray reflected from a surface is governed by the law: angle of incidence equals angle of reflection.

Problem 4. Show in two ways that a light ray incident on the parabola (1) along a line parallel to its axis will be reflected to the focus:

(Analytic Method) Demonstrate using Calculus and Trigonometry that a horizontal ray incident at $P$ and the focal line at $P$ make equal angles with either the tangent or the normal to the parabola at $P$.

(Synthetic Method) Demonstrate the same with only geometric arguments, using the characterization of the tangent and normal lines you derived in Problems 1 and 2.

4 Synthetic and Analytic Geometry

As a Cambridge undergraduate Isaac Newton learned Descartes’s coordinate geometry. He had to struggle through the book on his own, however, because at Cambridge in those days the Geometry syllabus was based on Euclid’s Elements. Later in his life, however, Newton called coordinate geometry “the Analysis of the Bunglers in Mathematricks,” and remarked,

Men of recent times, eager to add to the discoveries of the ancients, have united the arithmetic of variables with geometry. Benefiting from that, progress has been broad and far-reaching if your eye is on the profuseness of output, but the advance is less of a blessing if you look at the complexity of its conclusions. For these computations ... often express in an intolerably roundabout way quantities which in geometry are designated by the drawing of a single line.

Problem 5. Discuss Newton’s remarks in the light of your experience on Problem 4 and work in class. Answer the following questions: Which is more complex, the analytic method or the synthetic method? Which provides more insight when you solve a problem? Do you find that coordinate geometry expresses quantities in an “intolerably roundabout” way?