LOCAL ANALYTIC CONJUGACY OF SEMI-HYPERBOLIC MAPPINGS IN TWO VARIABLES, IN THE NON-ARCHIMEDEAN SETTING

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In this note, we consider locally invertible analytic mappings of a two-dimensional space over a non-archimedean field. Such a map is called semi-hyperbolic if its Jacobian has eigenvalues \( \lambda_1 \) and \( \lambda_2 \) so that \( \lambda_1 = 1 \) and \( |\lambda_2| \neq 1 \). We prove that two analytic semi-hyperbolic maps are analytically equivalent if and only if they are formally equivalent, applying a generalized version of an estimation scheme from our earlier work [A. Jenkins and S. Spallone, A \( p \)-adic approach to local dynamics: Analytic flows and analytic maps tangent to the identity, Ann. Fac. Sci. Toulouse Math. (6) 18(3) (2009) 611–634].

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1. Introduction

The basic questions of discrete dynamical systems surround the study of iterates of mappings defined in some set. For example, one can consider a global situation, where a domain \( S \) is fixed, and the goal is to understand the behavior of maps \( f : S \to S \). Here the totality of \( S \) is crucial in understanding their dynamics.

On the other hand, there is much to be learned from the local dynamics of maps. Rather than fixing a particular set \( S \) and studying the maps which send \( S \) to \( S \), we consider the dynamical properties of maps defined on a sufficiently small neighborhood of an interesting dynamical point, e.g. a fixed point \( P \). Let us say...
that $S$ is a subset of some vector space, and $P = 0 \in S$. In this setting, the actual neighborhood of 0 on which such a map is defined is mostly irrelevant (although geometric considerations may still play a large role in the theory). For this reason, one often considers the study of the dynamics of “germs” of mappings.

As an example, consider the set of germs of analytic functions $f$ fixing $0 \in \mathbb{C}$. Such germs may be written in the form:

$$f(z) = \sum_{n=m \geq 1} ^\infty a_n z^n.$$

A natural goal is the local reduction of such a mapping to a simpler “normal” form $f_0$, which is easier to study, yet retains all of the dynamical properties of the original function $f$. This is accomplished via a local change of variable, i.e. a map $h$ fixing 0 which conjugates $f$ to $f_0$ within some suitably small neighborhood $U$ of $0: h \circ f \circ h^{-1} = f_0$. Note that if such an analytic map exists, then obviously we have $h \circ f^n \circ h^{-1} = f^n_0$, and so all dynamical properties of $f$ are preserved. Moreover, depending on the regularity of $f$, more subtle data can be gained (for example, if $h$ is analytic, then it will preserve invariant analytic curves, etc.). We refer the reader to Abate [1] for a survey of this vast theory.

Before tackling analytic equivalence of germs of analytic mappings, it is often desirable to first determine the “formal” equivalence of such germs. Note that germs of analytic maps at 0 can be expressed via power series with no constant term. If we restrict ourselves to the set $G$ of (locally) invertible analytic germs, then $G$ forms a group. In fact, it is a subgroup of the group of invertible formal power series. We can thus consider a weaker relation: two analytic germs are called formally equivalent if they are conjugate in this larger group. The advantages of considering formal equivalence are numerous: while formal equivalence obviously does not guarantee analytic equivalence, it is easier to study, typically requiring only arithmetic operations. Because of this, formal theory within, e.g. $\mathbb{C}^n$ can be carried over to any field of characteristic 0. Moreover, a robust theory exists, with many “formal normal forms” (see Sec. 3 for some examples). The natural question thus arises: what can be said about two formally equivalent analytic germs $F$ and $G$?

The authors have turned to applying this formal theory in the non-archimedean case. The study of non-archimedean dynamics is an active area of research, encompassing both global and local results; see the works of Benedetto (e.g. [2]) and Rivera-Letelier (e.g. [10]). Rather than working in the topological fields $K = \mathbb{C}$ or $\mathbb{R}$, one takes $K$ to be the arithmetically-defined field $\mathbb{Q}_p$ or an extension. These are the non-archimedean fields of characteristic 0, and play a prominent role in number theory. However, our interest here is not arithmetic, but rather the gentler analytic theory that such fields provide. The norms on these fields satisfy the so-called ultrametric inequality

$$|x + y| \leq \max(|x|, |y|).$$
and as a consequence, a series $\sum a_n$ converges if and only if the terms $a_n$ converges 0. This makes convergence of power series particularly straightforward.

For a non-archimedean field $K$, the analytic theory is the same as the formal theory. That is, two convergent power series $f$ and $g$ which fix a point $P \in K$ are formally equivalent if and only if they are (locally) analytically equivalent. We refer the reader to [3, 6, 8] for the proofs of this fact in various cases and with various estimates. However, in several variables, this result is false. Our project is meant as an extension of the fundamental work of Herman and Yoccoz [6], which considered the case for which the maps are formally linearizable.

Suppose that the eigenvalues of the Jacobian $DF_0$ of $F$ are $\lambda_1, \ldots, \lambda_r$. The obstacle to the formal linearization of $F$ is resonance, which is a relation of the form:

$$\lambda_j - \lambda_1^{i_1} \lambda_2^{i_2} \cdots \lambda_r^{i_r} = 0. \quad (1.1)$$

Here for $1 \leq k \leq r$, the $i_k \geq 0$ and $\sum_k i_k \geq 2$. These differences arise as denominators in attempting to construct the formal maps which conjugate $F$ to its linear part. Herman and Yoccoz show that if the left-hand side of (1.1) stays away from 0, then $F$ is analytically linearizable. There are similar theorems in the complex case due to Siegel [12], Bryuno [4] and Yoccoz [16]. On the other hand, one also finds in [6] a wealth of examples of formally linearizable two-dimension maps which are not analytically linearizable, constructed by setting up these differences to be very small.

We have begun the project of hunting for species of analytic maps for which formal equivalence implies analytic equivalence. The resonant case is not covered by [6], and so many interesting families can be considered. For the case of invertible contracting germs in dimension two, “formal implies analytic” is proven in [15].

In this paper we offer another example of such a family. Let $K$ be a non-archimedean field, and let $F$ be an analytic map with fixed point 0 $\in K^2$. We will take $F$ to be semi-hyperbolic, in the sense that the eigenvalues at 0 are 1 and $\lambda$, where $|\lambda| \neq 0, 1$. Thus we write

$$F(x, y) = (x + O(2), \lambda y + O(2)), \quad (1.2)$$
given by two power series in the variables $x, y$ which are analytic at 0. This is one of the simplest cases of resonance, and we approach it with the same methods as the one-dimensional case of multiplier 1. Our main result is the following theorem.

**Theorem 1.1.** Let $|\lambda| \neq 0, 1$, and suppose that $F$ and $G$ are analytic mappings of the form (1.2). Then $F$ and $G$ are formally equivalent if and only if they are analytically equivalent.

This is actually a corollary to two normalization steps, which are of independent interest. First, we prove that each such map $F$ is analytically equivalent to its
Poincaré-Dulac (or “PD”) form

\[ F_0(x, y) = \left( x + \sum_{i=2}^{\infty} a_i x^i, \lambda y \left( 1 + \sum_{j=1}^{\infty} b_j x^j \right) \right). \]  (1.3)

Our method, which has its genesis in our one-variable paper [8] incorporates estimates into the PD-algorithm which are compatible with composition. This does not yet suffice to give Theorem 1.1, since the PD-form is far from being unique. For the second step, we turn to the earlier work of Jenkins [7], which formally reduces (1.3) to a certain polynomial form, which we refer to as the PDJ-form. More precisely, we have the following theorem.

**Theorem 1.2.** Let

\[ F_0(x, y_1, \ldots, y_n) = (f(x), \lambda_1 y_1(1 + g_1(x)), \ldots, \lambda_n y_n(1 + g_n(x))), \]  (1.4)

where \( \lambda_1, \ldots, \lambda_n \in K^\times \), \( f \) and \( g_i \) are analytic at 0, with \( f(x) = x + \rho x^m + O(x^{m+1}) \), with \( \rho \neq 0 \), and \( g_i(0) = 0 \). Write \( r_i(x) \) for the remainder of \( g_i(x) \) upon division by \( x^m \). Then, there is an analytic mapping \( H = H(x, y_1, \ldots, y_n) = (h(x), y_1 k_1(x), \ldots, y_n k_n(x)) \) with \( h \) tangent to the identity and \( k_i(0) = 1 \), and \( \mu \in K \), so that

\[ H \circ F_0 \circ H^{-1}(x, y_1, \ldots, y_n) \]

\[ = (x + \rho x^m + \mu x^{2m-1}, \lambda y_1(1 + r_1(x)), \ldots, \lambda y_n(1 + r_n(x))). \]

The PDJ-form is unique up to the action of an \((m - 1)\)-root of unity on the \( r_i(x) \). In this paper we review this reduction and prove that is actually an analytic conjugation, using another argument akin to that of [8]. The semi-hyperbolic case of Theorem 1.1 now follows easily. Again, this result is of independent interest; in the complex case, even maps of the special form (1.4) are not analytically equivalent to their PDJ-form [7]. We will discuss this further in Sec. 3.

We now describe the layout of this paper. Sec. 2 recalls the basics of non-archimedean fields and analytic functions. In Sec. 3 we present the theory of formal equivalence in the form that we need. We also discuss the manifold difficulties for semi-hyperbolic normalization when \( S = \mathbb{C}^2 \). The first normalization occurs in Sec. 4, where we prove that semi-hyperbolic maps are analytically equivalent to their PD-form. The second occurs in Sec. 5, yielding Theorem 1.2 and therefore Theorem 1.1. Some concluding remarks may be found in Sec. 6.

2. Preliminaries and Notation

2.1. Non-archimedean fields

A non-archimedean field \( K \) is a complete normed field, where the norm is non-archimedean.
Definition 2.1. Let $K$ be a field. A non-archimedean norm on $K$ is a map $|·|: K \to \mathbb{R}$ satisfying the following rules, for all $x, y \in K$:

(i) $|x| \geq 0$, $|x| = 0$ if and only if $x = 0$.
(ii) $|x + y| \leq \max(|x|, |y|)$.
(iii) $|xy| = |x||y|$.

In most of this paper (e.g. Sec. 4 onwards), $K$ is a non-archimedean field of characteristic 0. We will assume it is not discrete. The most basic examples of non-archimedean fields are the $p$-adic numbers $\mathbb{Q}_p$, defined as follows. Fix a prime $p \in \mathbb{Z}$, and consider the function $|m/n|_p = \left(\frac{1}{p}\right)^{\text{ord}_p(m) - \text{ord}_p(n)}$ on $\mathbb{Q}$ where $\text{ord}_p(n)$ is the exponent of $p$ in the prime factorization of $n$. Then $|·|_p$ is a non-archimedean norm on $\mathbb{Q}$, and the field of $p$-adic numbers $\mathbb{Q}_p$ is defined to be the topological completion of this normed field.

Balls in normed fields are defined in the usual way.

Definition 2.2. For $x_0 \in K$ and $\varepsilon > 0$, we define the “open” and “closed” balls centered at $x_0$ of radius $\varepsilon$ as

$$B(x_0, \varepsilon) = \{x \in K : |x - x_0| < \varepsilon\},$$
$$B^+(x_0, \varepsilon) = \{x \in K : |x - x_0| \leq \varepsilon\}.$$  

Let $\Delta = B^+(0, 1)$. The elements of $\Delta$ form a ring called the integers of $K$.

The following estimate will be critical to the proof of Theorem 1.2. For a reference, and a more leisurely introduction to non-archimedean analysis, see [11].

Proposition 2.3. Given a field $K$ of characteristic 0 with non-archimedean norm $|·|$, there is an $\alpha \in \mathbb{R}$ with $\alpha > 0$ so that for all natural numbers $n$, we have $|n!| \geq \alpha^n$.

For example, if $\mathbb{Q}_p \subseteq K$, then we may take $\alpha = 1/p$.

2.2. Analytic mappings

Write $\mathbb{N}$ for the set of nonnegative integers.

Definition 2.4. Given a vector $\vec{i} = (i, j) \in \mathbb{N}^2$, let

$$|\vec{i}| = i + j$$

Definition 2.5. Given a formal power series $f \in K[[x, y]]$, and a vector $\vec{i} = (i, j) \in \mathbb{N}^2$, write $[f]_{\vec{i}}$ for the coefficient of $x^iy^j$ in $f$. 

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Definition 2.6. Let \((a, b) \in K^2\) and let \(\varepsilon_1, \varepsilon_2\) be positive real numbers. Then the product of balls \(B(a, \varepsilon_1) \times B(b, \varepsilon_2)\) in \(K^2\) is called a polydisk. A power series \(f \in K[[x, y]]\) is analytic (at 0) if it converges in some polydisk in \(K^2\) containing 0.

As usual, geometric growth of coefficients implies analyticity.

Lemma 2.7. Let \(f \in K[[x, y]]\). Suppose that there is a number \(R > 0\) so that for all \(\vec{i}\) with \(|\vec{i}|\) sufficiently large, we have

\[
|\left[ f \right]_{\vec{i}}| \leq R^{|\vec{i}|}.
\]

Then \(f\) is analytic at 0.

Proof. In fact it converges on the polydisk \(B(0, \varepsilon) \times B(0, \varepsilon)\), when \(0 < \varepsilon < \frac{1}{R}\). \(\square\)

Definition 2.8. Let \(K[[x, y]]_0\) denote the set of formal power series \(f\) in \(r\) variables with zero constant term, i.e. with \(f(0) = 0\). Let \(n \geq 1\). Put

\[
I[n] = \{f \in K[[x, y]]_0 | \left[ f \right]_{\vec{i}} = 0 \text{ unless } |\vec{i}| \geq n\}.
\]

Definition 2.9. Let \(\mathcal{F} = \mathcal{F}_2\) be the set of formal maps \(F : K^2 \to K^2\) with \(F(0) = 0\). These are given in the usual way by pairs of power series in \(K[[x, y]]_0\). Let

\[
\mathcal{F}[n] = \{F \in \mathcal{F} | \pi_1 F, \pi_2 F \in I[n]\}.
\]

For \(F \in \mathcal{F}\), write \(DF_0\) for its linear part (the Jacobian at 0), and let \(\tilde{F} = F - DF_0\).

The expression \(\pi_k F\) above denotes the \(k\)th coordinate of \(F\). We also write \([f]_{\vec{i}}^k\) for the coefficient of \(x^i y^j\) in \(\pi_k F\), if \(\vec{i} = (i, j)\). Of course, \(F\) is analytic (at 0) if and only if each \(\pi_k F\) is analytic.

By the next lemma, whose proof we omit, we will often assume that \(\tilde{F}\) has integral coefficients. Write \(L_q\) for scalar multiplication by an element \(q \in K\).

Lemma 2.10. Let \(F \in \mathcal{F}\) be analytic. Then there is a \(q \in K^\times\) so that

\[
[L_q^{-1} \circ F \circ L_q]^k_{\vec{i}} \in \Delta,
\]

for \(k = 1, 2\) and for all \(\vec{i}\) with \(|\vec{i}| \geq 2\).

The following lemma will be useful later. Its proof is immediate.

Lemma 2.11. Let \(m \geq 2\). Suppose

\[
H_m(x, y) = \left( x + \sum_{i+j=m} c_{ij} x^i y^j, y + \sum_{i+j=m} d_{ij} x^i y^j \right).
\]
Let $G \in \mathcal{F}_2$ with $DG_0(x, y) = (\lambda_1 x, \lambda_2 y)$ for some $\lambda_1, \lambda_2 \in K$. Then for $i + j = m$ we have

$$[H_m \circ G^1]_{(i,j)} = [G^1]_{(i,j)} + c_{ij} \lambda_1^j \lambda_2^i,$$

and

$$[H_m \circ G^2]_{(i,j)} = [G^2]_{(i,j)} + d_{ij} \lambda_1^j \lambda_2^i.$$

For $n \geq 2$, $i + j = m$, and $k = 1, 2$ we have

$$[(\pi_k (H_m \circ G))^n]_{(i,j)} = [(\pi_k G)^n]_{(i,j)}.$$

3. Formal Equivalence

3.1. One variable

We recall here some of the formal theory in one variable, which holds for any field $K$ of characteristic zero. In the complex setting the formal theory has been known for some time, and is impossible to ascribe to a single source. The theory is entirely algebraic and needs only minor modifications when $K$ is not algebraically closed.

We will consider series tangent to the identity.

**Proposition 3.1.** Suppose that $f(x) = x + \rho x^m + \mu x^{2m-1} + O(x^{2m})$, with $\rho, \mu \in K$ and $\rho \neq 0$.

(i) $f$ is formally equivalent to $f_0(x) = x + \rho x^m + \mu x^{2m-1}$.

(ii) Suppose that $f$ is formally equivalent to another power series of the form $g(x) = x + \rho' x^n + \mu' x^{2n-1} + O(x^{2n})$. Then $m = n$, and there exists $c \in K^\times$ so that $c^{m-1} \rho' = \rho$ and $c^{2m-2} \mu' = \mu$.

Fix sets $R_j$ of coset representatives for $K^\times / (K^\times)^j$, so that all $\rho \in R_j$ have $|\rho| \leq 1$. For example, if either $K$ is algebraically closed or if $j = 1$, we may pick $R_j = \{1\}$. If $K = \mathbb{Q}_p$ with $p$ odd then we may pick $R_2 = \{1, \epsilon, p, \epsilon p\}$, where $\epsilon$ is not a square and $|\epsilon| = 1$.

**Definition 3.2.** We say that a map $f$ with multiplier one is in (rational) formal normal form if $f(x) = f_{m,\rho,\mu}(x) = x + \rho x^m + \mu x^{2m-1}$ with $\rho \in R_{m-1}$.

From Proposition 3.1, any map $f$ tangent to the identity is formally conjugate to a unique formal normal form. This choice is convenient for the proof of Theorem 1.2.

In one variable, the formal and analytic theories coincide, as seen in the following proposition.

**Proposition 3.3.** If $K$ is non-archimedean, then two germs at 0 of analytic maps on $K$ are analytically equivalent if and only if they are formally equivalent. In particular each is analytically equivalent to its formal normal form.
For the purposes of this paper, we combine this result with Proposition 3.1, and so any map which is tangent to the identity will be analytically conjugated to a polynomial form. This will be fundamental in our reductions of semi-hyperbolic maps in two variables. We refer the reader to [3, 6, 8] for proofs.

3.2. Poincaré–Dulac theory

This section gives a quick look at some of the formal normalizations of mappings $F \in \mathcal{F}$. We explain here how to eliminate terms in a power series mapping of the form

$$F(x_1, \ldots, x_n) = (\lambda_1 x_1 + O(2), \ldots, \lambda_n x_n + O(2)).$$

(3.1)

Recall the following definition.

Definition 3.4. Let $\lambda_i \in K$ for $i = 1, \ldots, n$. A resonance of the set $\{\lambda_1, \ldots, \lambda_n\}$ is a relation of the form

$$\lambda_j = \prod_{i=1}^{n} \lambda_i^{m_i},$$

(3.2)

where $m_i$ is a nonnegative integer for each $i$ and $\sum m_i \geq 2$.

For example, if $1 > |\lambda_1| \geq \cdots \geq |\lambda_n| > 0$, then only finitely many resonances can exist. On the other hand, note that any set of the form $\{1, \lambda_1, \ldots, \lambda_n\}$ will possess infinitely many resonances, regardless of the subset $\{\lambda_1, \ldots, \lambda_n\}$. In particular, we have that

(i) $1 = 1^n$ for all $n \geq 2$,

(ii) $\lambda_i = 1^n \lambda_i$ for all $n \geq 1$.

We say that the set $\{\lambda_1 = 1, \lambda_2, \ldots, \lambda_n\}$ is semi-nonresonant if the set $\{\lambda_2, \ldots, \lambda_n\}$ is nonresonant.

Let $F, G \in \mathcal{F}$ of the form (3.1). Our conjugating maps will take the form $H_n = Id + P_n(x)$, where $P_n$ is a homogeneous polynomial mapping of degree $n$, $n \geq 2$. Write $\Phi_n = H_n \circ \cdots \circ H_2$, and write $F_n = \Phi_n \circ F \circ \Phi_n^{-1}$, for $n \geq 2$, and $F_1 = F$. Each $H_n$ will be used to eliminate all terms of degree $n$ in $F$, except those which correspond to resonances in the eigenvalues. In order to accomplish this, we determine $H_m$ by the formulae (here, $m = \alpha_1 + \cdots + \alpha_n$):

$$[H_m]_{(\alpha_1, \ldots, \alpha_n)} = \frac{[F_{m-1}]_{(\alpha_1, \ldots, \alpha_n)}^{k} - [G]_{(\alpha_1, \ldots, \alpha_n)^{k}}}{\lambda_k - \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n}}.$$  

(3.3)

Of course, this algorithm breaks down if the denominator is zero, i.e. at any resonance of $\{\lambda_1, \ldots, \lambda_n\}$. Note that if no resonance relations exist, then the map may be formally linearized. If resonances exist, then this procedure will reduce the mapping $F$ to a form $F_0$ consisting of a sum of only resonant monomials, i.e. those
monomials with multidegrees corresponding to some resonance condition. We will refer to this map $F_0$ as the Poincaré–Dulac (or PD) form of $F$. Note that, a priori, it is not clear that $F_0$ is actually an analytic mapping if infinitely many resonant conditions exist (e.g. if $\lambda_1 = 1$).

We remark that any coefficient of $H_n$ which is not used in the simplification (i.e. any coefficient of a term whose multidegree corresponds to a resonance) can be considered a “free term”, and so if a resonance exists, then the conjugating map taking $F$ to its PD-form is not unique. We will take these free terms to be zero in what follows.

Main Example. Let us suppose that $n = 2$, and that $K$ is any field of characteristic 0. Let $F$ be a formal map of the form:

$$F(x, y) = (x + O(2), \lambda y + O(2)),$$

with $\lambda$ not a root of unity. As mentioned at the beginning of this subsection, infinitely many resonances are present in this case. Thus, our map $G$ will take the form of an infinite power series with nonzero coefficients $[G^1_{1, 0}]$ and $[G^2_{0, 1}]$. In general, the Poincaré–Dulac algorithm yields a family of formal conjugating maps which is dependent on infinitely many parameters, each corresponding to a choice of resonant monomial. Note again that, a priori, no analyticity can be assumed for the formal normal form $G$, even when the original map $F$ is analytic.

3.3. Further normalizations of Poincaré–Dulac forms

For semi-non-resonant maps, the Poincaré–Dulac algorithm leaves infinitely many terms in each component. However, such maps may be further normalized formally to a polynomial form, as was done by Jenkins [7].

To fix ideas, assume that our map $F$ takes the form

$$F(x, y) = (f(x), \lambda y(1 + g(x))).$$

(3.4)

Here $\lambda \neq 0$, $f$ is tangent to the identity, and $g$ is a formal power series with $g(0) = 0$.

We begin by reducing $\pi_1 F = f$ to its (rational) formal normal form. From Proposition 3.1, choose $\rho \in R_j$, $\mu \in K$, and a power series $h(x) = x + \cdots$, so that if

$$H_0(x, y) = (h(x), y),$$

then

$$H_0^{-1} \circ F \circ H_0 = F_1,$$

where

$$F_1(x, y) = (f_{m, \rho, \mu}(x), \lambda y(1 + g_1(x))).$$

(3.5)

Here $g_1 = g \circ h \in K[[x]]_0$. Thus, we may assume that $\pi_1 F = f_{m, \rho, \mu}$.
Next is the significant step of reducing the second component to \( \lambda y (1 + r(x)) \), where \( r(x) \) is the remainder of \( g(x) \) upon division by \( x^m \). One does this with maps of the form \( K(x, y) = (x, yk(x)) \), where \( k(0) = 1 \). This algorithm will be described in the proof of Theorem 1.2.

3.4. Uniqueness of normal forms

In this section we treat the uniqueness of the formal normal forms of this paper.

**Lemma 3.5.** Let \( f(x) = x + Ax^m + O(x^{2m-1}) \), with \( A \neq 0 \), and suppose that \( h \in K[[x]]_0 \) is an invertible series which centralizes \( f \). Then \( h(x) \equiv \zeta x \mod x^m \), where \( \zeta \) is an \((m-1)\)-root of unity.

**Proof.** Let \( h(x) = \sum_{n \geq 1} a_n x^n \), and with \( h \circ f = f \circ h \). Equating the \( m \)th coefficients gives

\[
a_1 A + a_m = Aa_1^m + a_m,
\]

and so \( a_1 = \zeta \) is an \((m-1)\)-root of unity.

Next, we prove by induction on \( 2 \leq n < m \) that \( a_n = 0 \). Thus, writing \( h(x) = \zeta x + a_n x^n + \ldots \), one computes that

\[
\[h \circ f\]_{m + (n-1)} = a_{m + (n-1)} + na_n A
\]

and

\[
[f \circ h]_{m + (n-1)} = a_{m + (n-1)} + Ama_n.
\]

This implies that \( a_n = 0 \), as desired. \( \square \)

Note that if \( \zeta \) is an \((m-1)\)-root of unity, then the linear map \( L_{\zeta}(x) = \zeta x \) does indeed centralize \( f \).

**Definition 3.6.** A map

\[
F(x, y_1, \ldots, y_n) = (f(x), \lambda_1 y_1 (1 + r_1(x)), \ldots, \lambda_n y_n (1 + r_n(x)))
\]

is in PDJ-normal form if

(i) \( \{\lambda_1, \ldots, \lambda_n\} \) is nonresonant.
(ii) \( f = f_{m, \rho, \mu} \) is in (rational) formal normal form.
(iii) For all \( i \), each \( r_i \) is a polynomial with \( \deg r_i < m \) and \( r_i(0) = 0 \).

A PDJ-normal form is almost a formal invariant.

**Proposition 3.7 (Uniqueness of PDJ-normal Form).** Let \( F \) and \( G \) be maps in PDJ-normal form, with

\[
F(x, y_1, \ldots, y_n) = (f(x), \lambda_1 y_1 (1 + r_1(x)), \ldots, \lambda_n y_n (1 + r_n(x))), \quad \text{and}
\]

\[
\tilde{F}(x, y_1, \ldots, y_n) = (\tilde{f}(x), \lambda_1 y_1 (1 + \tilde{r}_1(x)), \ldots, \lambda_n y_n (1 + \tilde{r}_n(x))).
\]

Suppose that \( F \) and \( G \) are formally equivalent. Then \( f = \tilde{f} \), and there is an \((m-1)\)-root of unity \( \zeta \) so that \( r_i(x) = \tilde{r}_i(\zeta x) \) for all \( i \).
Proof. Suppose that \( \Phi \) is an invertible map with
\[
\Phi \circ F = G \circ \Phi. \tag{3.6}
\]
By [7, Lemma 2.1], we may write \( \Phi(x, y) = (h(x), y_1 k_1(x), \ldots, y_n k_n(x)) \) for an invertible power series \( h \in K[[x]]_0 \) and \( k_i \in K[[x]] \), with \( k_i(0) \neq 0 \). The first component of (3.6) gives \( h \circ f = \tilde{f} \circ h \). Since \( f \) and \( \tilde{f} \) are in normal form, we have \( f = \tilde{f} \) and \( h \) centralizes \( f \). Say \( f(x) = x + \rho x^m + \mu x^{m-1} \), with \( \rho \in R_{m-1} \). The \((i+1)\)-component of (3.6) gives
\[
(1 + r_i(x))(k_i \circ f)(x) = k_i(x)(1 + (\tilde{r}_i \circ h)(x)).
\]
Using Lemma 3.5 and reading this equation modulo \( x^m \) gives
\[
(1 + r_i(x))k_i(x) \equiv k_i(x)(1 + \tilde{r}_i(\xi x)).
\]
Since \( k_i(0) \neq 0 \), we may multiplicatively invert \( k_i(x) \) modulo \( x^m \). It follows that \( r_i(x) \equiv \tilde{r}_i(\xi) \) modulo \( x^m \). Since \( \deg(r_i(x)), \deg(\tilde{r}_i(\xi)) < m \), it follows that they are equal.

Corollary 3.8. Let \( F \) and \( \tilde{F} \) be maps in PDJ-normal form. If \( F \) and \( \tilde{F} \) are formally equivalent, then they are analytically equivalent.

Proof. Since they are formally equivalent, they have the same eigenvalues. Conjugating by a linear transformation in the form of a permutation matrix, we may assume that \( F \) and \( G \) take the form in the proposition above. Therefore there is an \((m-1)\)-root of unity \( \xi \) so that \( r_i(x) = \tilde{r}_i(\xi x) \) for all \( i \). Let \( H(x, y_1, \ldots, y_n) = (\xi x, y_1, \ldots, y_n) \). It is easy to see that
\[
H \circ F = \tilde{F} \circ H.
\]

3.5. The complex case

This paragraph is devoted to a brief look at this problem in the complex case. The problem of semi-hyperbolic local analytic transformations has been studied by a number of mathematicians (see, e.g. [5, 7, 13, 14]).

We begin with the Poincaré–Dulac theory. Given a map
\[
F(x, y) = (x + O(2), \lambda y + O(2))
\]
where \( 0 < |\lambda| < 1 \), Ueda [13] has proven that such a map is analytically conjugated to a simpler analytic form for such maps in a neighborhood of the origin. (In higher dimensions, Hakimi [5] has given a similar normal form.) This form is given by the following: for any \( i, j \), there is an analytic change of variable \( H \) taking \( F \) to the form
\[
F_0(x, y) = \left( p(x) + \sum_{m=i+1}^{\infty} a_m(y)x^m, q(x) + \sum_{n=j+1}^{\infty} b_n(y)x^n \right).
\]
Here, \( p(x) \) is a polynomial in \( x \) of degree no greater than \( i \) which is tangent to the identity, \( q(x) \) is a polynomial in \( x \) of degree no greater than \( j \), and \( a_m, b_n \) are
locally-analytic functions in \( y \) defined near 0. Note that this is significantly more general than the PD-form of \( F \), in which the functions \( a_m \) are constant, and the functions \( b_n \) are linear monomials. A full reduction to the PD-form, however, is not always possible; Ueda gives a discussion which shows that the intertwining maps generally do not converge. Many of the ideas have their seed in the one-variable work of Voronin, where it is also known that formal conjugacy is significantly weaker than analytic conjugacy.

Even in the specialized case
\[
F(x, y) = (f(x), \lambda y(1 + g(x))),
\]
(3.7)
it is not possible to further reduce \( F \) to its PDJ-form. In fact, if one takes the one-variable theory of Voronin as known, this can be seen strictly by analyzing the formal theory. Jenkins [7] has shown that if \( \lambda \) is not a root of unity, then any formal map conjugating two maps of the form (3.7) must in fact take the form
\[
H(x, y) = (h(x), yk(x)),
\]
where \( h(0) = 0, k(0) \neq 0 \). In particular, when one conjugates \( f \) by \( H \), the first component yields \( h \circ f \circ h^{-1} \). Hence, any analytic theory in two variables depends on the corresponding theory in one variable. Further obstructions can also be identified even when analytic equivalence exists in the first component.

4. Analytic Equivalence of Semi-Hyperbolic Maps

Now consider mappings on \( K^2 \), with \( K \) non-archimedean as in Sec. 2.1. We prove now that a semi-hyperbolic mapping
\[
F(x, y) = (x + O(2), \lambda y + O(2)),
\]
(4.1)
with \( |\lambda| \neq 0, 1 \), is analytically equivalent to its PD-form
\[
F_0(x, y) = (f(x), \lambda y(1 + g(x))).
\]
(4.2)

We assume that \( F \) is an analytic mapping of two variables fixing the origin whose eigenvalues at 0 are 1 and \( \lambda \) satisfying \( 1 < |\lambda| \). (For the case \( 0 < |\lambda| < 1 \), one may consider the inverse \( F^{-1} \).) Using Lemma 2.10, we assume that \( \tilde{F} \) (the higher degree terms of \( F \)) has integer coefficients. Thus, we write
\[
F(x, y) = \left( x + \sum_{j+k=2} a_{jk} x^j y^k, \lambda y + \sum_{j+k=2} b_{jk} x^j y^k \right),
\]
(4.3)
where \( a_{ij}, b_{ij} \in \Delta \) for all \( (i, j) \) with \( i + j \geq 2 \).

Our goal is to show that the formal techniques of the Poincaré-Dulac theory yield both analytic normal forms as well as analytic intertwining maps. The PD-form of \( F \) has the form \( F_0(x, y) = (f(x), \lambda y(1 + g(x))) \); let us write this as
\[
F_0(x, y) = \left( x + \sum_{j=2} a_0^j x^j, \lambda y \left( 1 + \sum_{k=1} b_0^k x^k \right) \right).
\]
The formal conjugating map $H$ is uniquely determined if we assume that it is tangent to the identity, and that $\pi_1 H$ has no term of the form $c_0 x^i$, and $\pi_2 H$ has no term of the form $d_{ij} x^i y^j$.

**Definition 4.1.** Let $\mathfrak{G}$ be the set of $G \in \mathcal{F}$ so that for $m + n \geq 2$, and $k = 1, 2$, we have

$$\lambda^{\max(1, n)}[G]_{(m, n)}^k \in \Delta.$$ 

We leave the proof of the following to the reader.

**Proposition 4.2.** $\mathfrak{G}$ forms a group under composition.

The PD-form $F_0$ adds a complication to the matter; its construction is interlaced with the construction of $\Phi$. We will inductively see that its coefficients are integral by the next lemma. Recall that we construct $H_m$ and $\Phi_m$ for $m \geq n$ so that $\Phi_m \circ F = F_0 \circ \Phi_m$ modulo $\mathcal{F}[m + 1]$.

**Lemma 4.3.** Suppose that $F$ is written in the form (4.3) so that $\tilde{F}$ has integer coefficients. Suppose also that $\Phi_m \in \mathfrak{G}$. Then $\tilde{F}_0$ has integer coefficients modulo $\mathcal{F}[m + 1]$.

**Proof.** Since $\Phi_m^{-1} \in \mathfrak{G}$, we know in particular that $\lambda \Phi_m^{-1}$ has integer coefficients. It follows that $(F \circ \Phi_m^{-1})(x, y) = (x, \lambda y) + G(x, y)$, where $G$ has integer coefficients. Since $\Phi_m \in \mathfrak{G}$, we have $\Phi_m(x, \lambda y) = (x, \lambda y)$ is integral. Thus, the function $(\Phi_m \circ F \circ \Phi_m^{-1})(x, y) = (x, \lambda y)$ has integer coefficients. Since this function agrees with $F_0$ modulo $(m + 1)$-degree terms, the result is proved. \qed

**Proposition 4.4.** Let $F$ be an analytic germ of the form (4.3) with $a_{ij}$ and $b_{ij}$ integers. Let $H_m$ and $F_0$ be defined as before for all $m \geq n + 1$. Then $H_m \in \mathfrak{G}$, and $\tilde{F}_0$ has integer coefficients.

**Proof.** We induct on $m$. Let us first write

$$H_{m+1}(x, y) = \left( x + \sum_{i+j=m+1} c_{ij} x^i y^j, y + \sum_{i+j=m+1} d_{ij} x^i y^j \right).$$

Recall from the formal algorithm that we choose $c_{m+1,0} = d_{m,1} = 0$; this is important for the analysis that follows. By the formal theory, we have $\Phi_{m+1} \circ F \equiv F_0 \circ \Phi_{m+1}$ modulo $\mathcal{F}[m + 2]$ is satisfied. We will consider the terms of total degree $m + 1$ on each side of this equation.

We start by looking at the first components. For any $(i, j)$ with $i + j = m + 1$ and $j \neq 0$ (since we have already defined $c_{m+1,1} = 0$), Lemma 2.11 yields

$$[\Phi_{m+1} \circ F]_{(i, j)}^1 = [\Phi_m \circ F]_{(i, j)}^1 + c_{ij} \lambda^j,$$  \hspace{1cm} (4.4)
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and

$$[F_0 \circ \Phi_{m+1}]^1_{(i,j)} = [\Phi_m]^1_{(i,j)} + \sum_{k=2}^m a^0_k \left[ (\pi_1 \Phi_m)^k \right]_{(i,j)} + c_{ij}. \tag{4.5}$$

Equating the two sides gives

$$c_{ij} (1 - \lambda^j) = [\Phi_m \circ F]^1_{(i,j)} - [\Phi_m]^1_{(i,j)} - \sum_{k=2}^m a^0_k \left[ (\pi_1 \Phi_m)^k \right]_{(i,j)}. \tag{4.6}$$

Since $H_i \in \mathfrak{S}$ for $i < m + 1$ by hypothesis, we have $\Phi_m \in \mathfrak{S}$; in particular its coefficients are integral. Moreover, $\Phi(x, \lambda y) - (x, \lambda y)$ is integral. By Lemma 4.3, we have $a^0_j \in \Delta$ for $j \leq m$. Thus the right-hand side of (4.6) is an integer, and it follows that

$$\lambda \max(1, j) c_{ij} \in \Delta. \tag{4.7}$$

The proof for the second component is similar. Computing the $(i, j)$-terms for $i + j = m + 1$, with $i \neq m$, we have the equations

$$[\Phi_{m+1} \circ F]^2_{(i,j)} = [\Phi_m \circ F]^2_{(i,j)} + d_{ij} \lambda. \tag{4.8}$$

and

$$[F_0 \circ \Phi_{m+1}]^2_{(i,j)} = \lambda [\Phi_m]^2_{(i,j)} + \lambda \sum_{k=1}^m b^0_k \left[ (\pi_2 \Phi_m)(\pi_1 \Phi_m)^k \right]_{(i,j)} + d_{ij} \lambda. \tag{4.9}$$

Setting the two equal to one another we obtain

$$d_{ij} (\lambda - \lambda^j) = [\Phi_m \circ F]^2_{(i,j)} - \lambda [\Phi_m]^2_{(i,j)} - \lambda \sum_{k=1}^m b^0_k \left[ (\pi_2 \Phi_m)(\pi_1 \Phi_m)^k \right]_{(i,j)} \tag{4.10}$$

We argue that the right-hand side is again integral. The first term is integral as in the previous argument. Let $\phi_k = \pi_k \Phi_m$. For $k = 1, 2$, the polynomials $\lambda \phi_k$ have integral coefficients. The only possible non-integral term of degree $(i, j)$ in

$$\lambda(y + \phi_2)(x + \phi_1)^k,$$

then, is for $k = m$ and thus $(i, j) = (m, 0)$. However, we have already defined $d_{m1} = 0$, so the estimate holds trivially in this case. (Notice the left-hand side is 0.) It follows that

$$\lambda \max(1, j) d_{ij} \in \Delta. \tag{4.11}$$

Combining (4.7) and (4.11) gives the proposition.

**Proposition 4.5.** If $F \in \mathcal{F}_2$ has eigenvalues $\lambda$ and 1 with $\lambda$ not a root of unity, then $F$ is analytically equivalent to its PD-normal form.

**Proof.** By the previous proposition, we have $H_m$ and therefore $\Phi_m \in \mathfrak{S}$ for all $m$. It follows that $\Phi$ and $\Phi^{-1}$ lie in $\mathfrak{S}$. In particular, the coefficients of these functions are integers. Thus $F$ is analytically conjugate to its normal form.
5. Further Normalization

We have analytically normalized a given semi-hyperbolic map to the form

$$F_0(x, y) = (f(x), \lambda y(1 + g(x))), \quad (5.1)$$

where $f$ is a one-variable analytic map which is tangent to the identity, and $g$ is an analytic map satisfying $g(0) = 0$. We now turn to further normalization, following the theory outlined in Sec. 3.3. The result will be a polynomial normal form.

This reduction step is not appreciably more difficult in higher dimensions, so we describe the general reduction. Fix $\lambda_1, \ldots, \lambda_n \in K^\times$; they may or may not possess resonance. Consider an analytic function of the form

$$F(x, y_1, \ldots, y_n) = (f(x), \lambda_1 y_1(1 + g_1(x)), \ldots, \lambda_n y_n(1 + g_n(x))), \quad (5.2)$$

where $f$ is a one-variable analytic map tangent to the identity and $g_i \in K[[x]]_0$. By Proposition 3.3, there is a locally-analytic map $h = h(x)$ fixing 0 and $h^{-1}(x) = f_{m,\rho,\mu}$, and so by conjugating $F$ in (5.2) by the change of variable

$$H(x, y_1, \ldots, y_n) = (h(x), y_1, \ldots, y_n),$$

we may assume that $f = f_{m,\rho,\mu}$.

Let $L$ be the diagonal map with eigenvalues $(a, 1, \ldots, 1)$. An easy check shows that

$$(L \circ F \circ L^{-1})(x, y_1, \ldots, y_n)$$

$$= (af(x/a), \lambda_1 y_1(1 + g_1(x/a)), \ldots, \lambda_n y_n(1 + g_n(x/a))).$$

Thus, by choosing $a$ with sufficiently large norm, we may assume that the coefficients of $f$ and $g_i$ are small. In particular, we will assume that all of these coefficients lie in $\Delta$. Our goal is to conjugate $F$ to the reduced (PDJ) form

$$F_r(x, y_1, \ldots, y_n) = (f_{m,\rho,\mu}, \lambda y_1(1 + r_1(x)), \ldots, \lambda y_n(1 + r_n(x))), \quad (5.3)$$

where $r_i(x)$ is the remainder of $g_i(x)$ upon division by $x^m$.

The conjugation will be by an $a$ priori formal map

$$\Gamma = \lim_{n \to \infty} \Gamma_n = \lim_{n \to \infty} (J_n \circ J_{n-1} \circ \cdots \circ J_1),$$

where

$$J_i(x, y_1, \ldots, y_n) = (x, y_1(1 + c_{i,1}x^1), \ldots, y_n(1 + c_{i,n}x^i)).$$

The chosen form of the conjugating map is useful. First, note that for any $i = 1, 2, \ldots$, the inverse $J_i^{-1}$ (as well as the inverse $\Gamma_i^{-1}$) is easily determined. Moreover, when conjugating any $F$ of the form (5.2), the coefficients $c_{i,j}$ are used to reduce the analytic function $\pi_j F$ for $j = 1, \ldots, n$. The upshot is that we can isolate and eliminate terms of the functions $g_i$ individually, and thus, for ease of notation and with no loss of generality, we will assume from now on that $n = 1$, i.e. that the map $F = F(x, y)$.  

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Theorem 1.2 will be proven once we ascertain that $\Gamma$ is analytic, which is equivalent to proving that the infinite product $\prod_{i=1}^{\infty}(1 + c_i x^i)$ is analytic. The coefficients $c_i$ are determined inductively. As above, put $\Gamma_n = J_n \circ \cdots \circ J_1$, and let $\Gamma_0$ be the identity. We choose $c_n$ so that

\[ [F \circ \Gamma_n - \Gamma_n \circ F]_{(m+n-1,1)}^2 = 0. \]

We will inductively show that $n! \rho^n c_n \in \Delta$. This estimate will also yield a similar estimate for $\Gamma_n$, as the following lemma demonstrates.

Lemma 5.1. Suppose that $c_i \in K$ satisfy $i! \rho^i c_i \in \Delta$ for $1 \leq i \leq k$. Write

\[ \prod_{i=1}^{k}(1 + c_i x^i) = 1 + \sum_{j \geq 1} A_j x^j. \]

Then we have $n! \rho^n A_n \in \Delta$ for all $n \geq 1$.

Proof. We can write $A_n = \sum_{\underline{i}=n} \alpha_{\underline{i}} c_{\underline{i}}$, where $c_{\underline{i}} = c_{i_1} \cdots c_{i_l}$ and $\alpha_{\underline{i}} \in \mathbb{Z}$. By hypothesis,

\[ i_1! \cdots i_l! \rho^{|\underline{i}|} c_{\underline{i}} \in \Delta. \]

Since the quotient $\frac{n!}{i_1! \cdots i_l!}$ is an integer, we have $n! \rho^n c_{\underline{i}} \in \Delta$, and the lemma follows.

Proof of Theorem 1.2. We assume that $F$ is in PD-form (5.1), and that the coefficients of $f$ and $g$ are each integral. We now reveal how the coefficients $c_n$ are formed and inductively prove that they satisfy the estimate $n! \rho^n c_n \in \Delta$ for all $n$. By Lemma 5.1 and Proposition 2.3, these estimates will ultimately ensure that the function $\Gamma$ is analytic.

We consider the congruence

\[ \pi_2(\Gamma_n \circ F) \equiv \pi_2(F_r \circ \Gamma_n) \mod x^{m+n}, \]

for $n \geq 0$. This condition governs the choice of $c_n$.

Since $g$ and $r$ are congruent mod $x^m$, (5.4) holds for $n = 0$. To determine $c_{n+1}$, we look at the second components of

\[ J_{n+1} \circ \Gamma_n \circ F \quad \text{and} \quad F_r \circ J_{n+1} \circ \Gamma_n. \]

We choose $c_{n+1}$ so that the following equality holds, mod $x^{m+n+1}$:

\[ (1 + g(x))(1 + c_{n+1}(f_{m,\rho,\mu}(x))^{n+1})(1 + (\alpha \circ f_{m,\rho,\mu})(x)) = (1 + r(x))(1 + c_{n+1} x^{n+1})(1 + \alpha(x)). \]

Here $\alpha(x) = \sum_{j \geq 1} A_j x^j$ is defined by $\Gamma_n(x, y) = (x, y(1 + \alpha(x)))$. 

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We expand both sides of (5.5). After subtracting the left side from the right, and considering only the \((m + n)\)-degree terms, we solve for \(c_{n+1}\). First, note that
\[
(f_{m, \rho, \mu}(x))^{n+1} \equiv x^{n+1} + (n + 1) \rho x^{m+n} \mod x^{m+n+1}.
\]
Because of this, we have
\[
[c_{n+1}(f_{m, \rho, \mu}(x))^{n+1}]_{m+n} = (n + 1) \rho c_{n+1}.
\]
In addition,
\[
[g(x)(f_{m, \rho, \mu}(x))^{n+1} - r(x)x^{n+1}]_{m+n} = 0,
\]
and since \(g(x) \equiv r(x) \mod x^{m}\). Since \(\alpha\) has no constant term, we have
\[
(\alpha \circ f_{m, \rho, \mu})(x) \equiv \alpha(x) \mod x^{m}.
\]
Thus,
\[
[c_{n+1}(f_{m, \rho, \mu}(x))^{n+1}(\alpha \circ f_{m, \rho, \mu})(x) - c_{n+1}x^{n+1}\alpha(x)]_{m+n} = 0,
\]
By combining the above ideas, we conclude
\[
[c_{n+1}g(x)f_{m, \rho, \mu}(x)^{n+1}(\alpha \circ f_{m, \rho, \mu})(x) - c_{n+1}r(x)x^{n+1}\alpha(x)]_{m+n} = 0.
\]
Therefore the \((m + n)\)-coefficient obtained from (5.5) is the sum of \((n + 1) \rho c_{n+1}\) and
\[
[r(x) + \alpha(x) + r(x)\alpha(x) - (g(x) + (\alpha \circ f_{m, \rho, \mu})(x) + g(x)(\alpha \circ f_{m, \rho, \mu})(x))]|_{m+n}.
\]
(5.6)

At this point, we have completed the formal theory; the \((m + n)\)-coefficient must be 0, and this determines \(c_{n+1}\) uniquely for each \(n \geq 0\). Note that \(c_{n+1}\) is well-defined for each \(n\), as it depends only on a finite number of terms.

To settle analyticity, we estimate this coefficient. In light of Lemma 5.1, we will prove the estimate
\[
|(n+1)!c_{n+1}| \leq 1.
\]
(5.7)

Since (5.6) is equal to \(-(n+1) \rho c_{n+1}\), we must show that the product of \(n! \rho^n\) with (5.6) is integral.

We will prove this inductively. The reasoning is elementary and mostly combinatorial. We will estimate pieces of (5.6) individually; the estimate will hold for the sum of these pieces because our norm is non-archimedean.

For example, note that the difference \((r(x) - g(x))\) certainly satisfies our estimate, since our coefficients are integral, and both \(n! \) and \(\rho\) are integral. We will break the remainder of the argument into two steps in order to estimate the remaining terms of (5.6).
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**Step 1.** Let us consider here the difference \( \alpha \circ f_{m,\rho,\mu}(x) - \alpha(x) \). We can write

\[
(\alpha \circ f_{m,\rho,\mu})(x) - \alpha(x) = \sum_{j \geq 1} A_j ((x + \rho x^m + \mu x^{2m-1})^j - x^j).
\]

We will expand the powers on the right-hand side as follows: consider each of the terms \( (\alpha \circ f_{m,\rho,\mu})^j x^a (\rho x^m)^b (\mu x^{2m-1})^c A_j \).

Since we are only considering the coefficient of the \((m + n)\)-degree term, we must have

\[
a + b + c = j, \tag{5.8}
\]

and

\[
a + mb + (2m - 1)c = m + n. \tag{5.9}
\]

Furthermore, we see that \( b \) and \( c \) are not simultaneously 0, or else the term is trivial. Lemma 5.1 yields

\[
j! \rho^j A_j \in \Delta. \tag{5.10}
\]

There are two possibilities to consider: either \( j = n + 1 \) or \( j \leq n \). If \( j \leq n \), then it is clear that \(|\rho^n| \leq |\rho^j|\) and that \(|n!| \leq |j!|\). Thus, because \( A_j \) satisfies (5.10), it immediately satisfies the stronger estimate

\[
|n! \rho^n A_j| \leq 1.
\]

On the other hand, if \( j = n + 1 \), then we must have \((a, b, c) = (n, 1, 0)\). This triple corresponds to the single term \((n + 1)\rho A_{n+1} x^{m+n}\). From Lemma 5.1, we have

\[
(n + 1)! \rho^{n+1} A_{n+1} = n! \rho^n ((n + 1)\rho A_{n+1}) \in \Delta.
\]

Thus, our estimate is satisfied.

**Step 2.** We will now consider the difference \( r(x)\alpha(x) - g(x)(\alpha \circ f_{m,\rho,\mu})(x) \). By definition of \( r \), we have that \( g(x) = r(x) + \tilde{g}(x) \), where \( \tilde{g}(x) = O(x^m) \). Using this, we may rewrite this difference as

\[
r(x)\alpha(x) - r(x)(\alpha \circ f_{m,\rho,\mu})(x) - \tilde{g}(x)(\alpha \circ f_{m,\rho,\mu})(x).
\]

We begin by analyzing the series \( \tilde{g}(x)(\alpha \circ f_{m,\rho,\mu})(x) \). The \((m + n)\)-degree coefficient of this series will be a sum of terms of the form

\[
[g]_d [\alpha \circ f_{m,\rho,\mu}]_{m+n-d} \tag{5.11}
\]

Since \([g]_d = [\tilde{g}]_d\) for all \( d \geq m \), each term in (5.11) is in turn a sum of integer multiples of terms of the form

\[
[g]_d x^d A_j x^a (\rho x^m)^b (\mu x^{2m-1})^c,
\]

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where \([g]_d\) is integral, and \(d \geq m\). We also have
\[
a + b + c = j, \tag{5.12}
\]
but now
\[
a + mb + (2m - 1)c + d = m + n. \tag{5.13}
\]

We use the same techniques as in Step 1. Since \([g]_d\) is an integer, we need only to show
\[
n!p^\alpha(r A_j) \in \Delta. \tag{5.14}
\]
Again, we already have the estimate (5.10) for \(A_j\). Using the fact that \(d \geq m\), we can conclude that \(j \leq n\), and so our estimate follows as in Step 1.

In similar fashion, we consider
\[
r(x)\alpha(x) - r(x)(\alpha \circ f_{m,p,\mu})(x). \tag{5.15}
\]
Writing this out, we obtain a sum of terms which are products of the form
\[
\left[\sum_{i=1}^{m-1} a_i x^i\right]_d \left[\sum_{n \geq 1} A_j((x + \rho x^m + \mu x^{2m-1})^j - x^j)\right]_{m+n-d}.
\]
We look at all expansions of the above product, each of which take the form
\[
a_d x^d A_j x^a (\rho x^m)^b (\mu x^{2m-1})^c. \tag{5.16}
\]
Again
\[
a + b + c = j,
\]
and
\[
a + mb + (2m - 1)c + d = m + n.
\]
If \(d = 0\), the expansion is necessarily trivial. On the other hand, if \(1 \leq d \leq m - 1\), then putting together the equalities above, we see that \(j \leq n\). From the estimate (5.10), the result now follows.

**Proof of Theorem 1.1.** Suppose that \(F\) and \(G\) are formally equivalent mappings of the form (1.2) with the same eigenvalues \(\lambda\) and 1, with \(|\lambda| \neq 0, 1\).

By the previous proposition, we may assume that \(F\) and \(G\) are in PDJ-normal form. By Corollary 3.8, two such maps are formally equivalent if and only if they are analytically equivalent.

6. Concluding Remarks

The theorem here is another instance of the formal theory coinciding with the analytic theory in the non-archimedean setting, a radical departure from the theory in \(\mathbb{C}^n\). Another two-dimensional resonant family to consider is the saddle-hyperbolic case. To explain, let \(F(x, y) = (\lambda_1 x + O(2), \lambda_2 y + O(2))\), where \(0 < |\lambda_1| < 1 < |\lambda_2|\). The resonant case is where \(\lambda_1^a \lambda_2^b = 1\) for some \(a, b \in \mathbb{N}\). For the sake of this...
discussion, let the field be \( \mathbb{Q}_2 \), and let \( \lambda_1 = 2 \) and \( \lambda_2 = 1/2 \). Following the Poincaré-Dulac algorithm, the formal normal form is easily determined; it takes the form

\[
F_0(x, y) = \left( 2x + x \sum_{m=1}^{\infty} a_m(xy)^m + \frac{1}{2} y + y \sum_{n=1}^{\infty} b_n(xy)^n \right).
\]

The problem is then simply put: is \( F \) analytically equivalent to \( F_0 \)?

Related to the saddle-hyperbolic case is the following problem: let

\[
F(x, y, z) = (x + O(2), \lambda_1 y + O(2), \lambda_2 z + O(2)),
\]

where \( |\lambda_i| \neq 0 \) for \( i = 1, 2 \), and \( \lambda_1, \lambda_2 \) are non-resonant. Again, one may follow the Poincaré-Dulac theory to produce the PD-form, and then further normalize to produce a PDJ-form which is polynomial. Further, the proof in Sec. 5 shows that, if the PD-form is analytic, then the further normalization will also be analytic. So, a natural question to ask is whether or not the initial PD-normalization is analytic.

The obstacle that we have run up against in these two scenarios has been finding the proper analogue of the group \( \mathcal{G} \) of Definition 4.1. That is, finding a good estimate for the conjugating functions. It may be that such maps are simply not analytically conjugated to their formal normal forms, thus providing other examples of formally conjugate maps which are not analytically equivalent. We hope to settle this on another occasion.

Finally, we remark that very little is known in the case where the characteristic of the field \( K \) is \( p > 0 \). Even finding formal normal forms for mappings on such \( K \) remains an interesting open problem. See [9] regarding one-dimensional analytic linearization. Since the theory of semihyperbolic maps in two dimensions relies fundamentally on the theory of one-dimensional maps with multiplier one, we have not seriously considered this theory in this context.

References


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[10] J. Rivera-Letelier, Dynamique des fonctions rationnelle sur des corps locaux,
1984).
[13] T. Ueda, Local structure of analytic transformations of two complex variables, I,
[14] T. Ueda, Local structure of analytic transformations of two complex variables, II,
[16] J.-C. Yoccoz, Théorème de Siegel, nombres de Bryuno et polynômes quadratique,